Steady State Heat Transfer using Galerkin Finite Element Method

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***Abstract*— The finite-element methods (FEMs) have become one of the major strategies for solving partial differential equations. Galerkin method is one of widely used FEM technique to solve the problems in heat transfer, fluid mechanics and mechanical systems. The objective of this paper is to perform error analysis for different types of heat flow vectors using Galerkin method for triangular elements. The heat flow vectors applied in this article include constant, exponential, sinusoidal and polynomial functions. On the basis of simulation results some guidelines has been suggested for each category of these heat flow vector focusing on method accuracy.**

***Keywords*— Finite-element methods; Galerkin method; Error Analysis; Triangular Elements**

I. Introduction

Partial differently equation or PDE is an equation involving multiple independent variables. PDEs are categorized in parabolic, hyperbolic and elliptic depending on the order of terms [1]. The elliptic PDEs model the steady state behavior of system. One example of such system is steady state heat distribution on rectangular plate; whose edges are held at particular temperature. In this type of PDE time is not a factor thus only x and y represent independent variables. Few problems of this category are focused in this article.

Finite Element Method or FEM is technique of mathematics for finding approximate numerical solution of PDE. Key concept of FEM is to subdivide the problem space in simpler and smaller parts. These parts are called finite elements. Latterly these small parts are combined to form the solution of original problem. Collocation, Sub domain and Galerkin Methods are known techniques of FEM.

In this paper a try has been made to analyze the error while solving the heat equation (elliptic PDE) for different heat flow vectors types by using the Galerkin method [2]. The general form of heat equation considered in this paper is

$$\Delta w+\chi(x,y)w=\varphi(x,y) \tag{1}$$

Here $\varphi(x,y)$ is the heat flow vector [3].

This paper is organized as follows: Related word is presented in next section II and section III describes the actual formulation of Galerkin method used in this paper. The section IV lists and explains the problem under consideration in this paper, while section V presents the discussion related result. Finally last section VI concludes the paper.

II. Related Work

Many authors have suggested different methods and techniques to solve PDEs using various FEM methods. Specifically to heat equation, in [4] author present numerical solution of one dimensional hear equation using Galerkin B-spline FEM. It uses quadratic B-Sline and results obtained are compared with exact solution. B-Splines are also used by different researchers for various problems in [5-7]. Similarly in [8] the authors presented a method for time dependent heat equation. The error resulted in Galerkin method due to change of heat transfer vector has not yet been studied as per our knowledge.

Galerkin method has been used to solve four different problems with different categories of heat flow vectors and errors are analyzed in various ways to obtain some conclusions.

III. The Galerkin Method

The elliptic PDE [9] is given as

$$\Delta w+\chi(x,y)w=\varphi(x,y) \tag{2}$$

The Dirichlet boundary condition for this problem can be given as

$$w(x,y)=\eta(x,y) \tag{3}$$

The residual is

$$r=\Delta w(x,y)+\chi(x,y)w(x,y)-\varphi(x,y) \tag{4}$$

Let ϕ_i be the piecewise linear functions over the rectangular region R. The orthogonality assumption [10] takes the form as

$$\iint_R(\Delta w+\chi w-\varphi)\phi_i dxdy=0 \tag{5}$$

Or

$$\iint_R(\Delta w+\chi w)\phi_i dxdy=\iint_R\varphi\phi_i dxdy \tag{6}$$

The above equation is called as the weak form of given elliptic equation.

Applying green's first identity [1] to the above weak form of equation and substituting the essence of FEM

$$w(x,y)=\sum_{i=1}^{n}\upsilon_i\phi_i(x,y) \quad (7)$$

We obtain the following form of given elliptic equation

$$\iint_R\left(\sum_{j=1}^{n}\upsilon_j\nabla\phi_j\right).\nabla\phi_i dxdy+ \iint_R\chi\left(\sum_{j=1}^{n}\upsilon_j\phi_j\right)\phi_i dxdy=\iint_R\varphi\phi_i dxdy \quad (8)$$

Factoring out the constant υ_j and solving for first m of them; yields a set of linear equations in the $\upsilon_1,\upsilon_2,\upsilon_3,\ldots,\upsilon_m$. In matrix form the equations is $A\upsilon=b$, the entries of A and b are given as

$$a_{ij}=\iint_R\nabla\phi_j\nabla\phi_i dxdy-\iint_R\chi\phi_j\phi_i dxdy \quad (9)$$

And

$$b_i=\iint_R\varphi\phi_i dxdy-\sum_{j=m+1}^{n}\upsilon_j\begin{bmatrix}\iint_R\nabla\phi_j\nabla\phi_i dxdy\\ -\iint_R\varphi\phi_j\phi_i dxdy\end{bmatrix} \quad (10)$$

The rectangular region R is composed of disjoint triangles. There are a total of $2(m+1)(n+1)$ triangles. The entries a_{ij} of system matrix A and b_i of load vector b are calculated by identities established for triangular base elements [11].

IV. Pproblems Solved

Four different problems were solved using above stated Galerkin Method. Each problem represents its own category to obtain a wide picture of error margin while using this FEM method. All the problems were analyzed for a unit square heating plate.

A. Exponential Heat Vector (P1)

This problem is taken from [12] the problem involves exponential function as heat vector, the problem equation is:

$$w_{xx}+w_{yy}=xe^y \quad (11)$$

With initial conditions as:

$$0<x,y<1$$
$$w(0,y)=0;w(1,y)=e^y \quad (12)$$
$$w(x,0)=x;w(x,1)=x.e$$

Having Analytical Solution as:

$$w(x,y)=xe^y \quad (13)$$

The solution is graphed as in Figure 1.

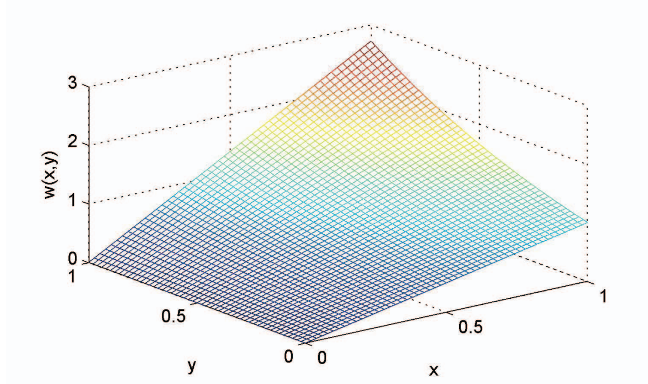

Figure 1. Analytical Solution of Problem P1

B. Zero Heat Vector / Laplace Equation (P2)

This problem is also taken from [12] the problem is standard Laplace equation with zero value heat vector. The problem is:

$$w_{xx}+w_{yy}=0 \quad (14)$$

With initial conditions as:

$$0<x,y<1$$
$$w(0,y)=0;w(1,y)=400x \quad (15)$$
$$w(x,0)=0;w(x,1)=400y$$

Having Analytical Solution as:

$$w(x,y)=400xy \quad (16)$$

The exact solution is presented in Figure 2.

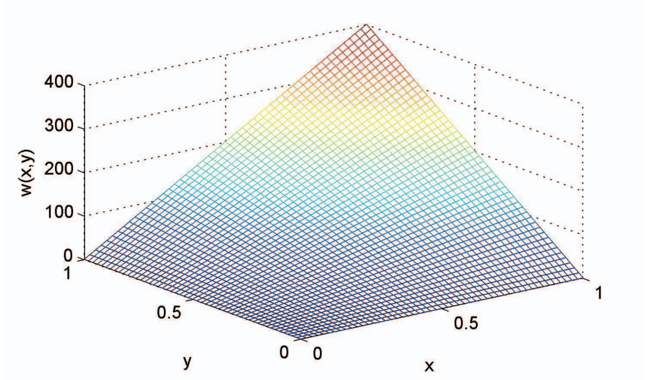

Figure 2. Analytical Solution of Problem P2

C. Polynomial Heat Vector (P3)

The third problem is also taken from [13] the problem involves polynomial heat vector. The problem is:

$$w_{xx} + w_{yy} = 12xy \tag{17}$$

With initial conditions as:

$$0 < x, y < 1$$
$$w(0,y) = 0; w(1,y) = 2y^3 \tag{18}$$
$$w(x,0) = 0; w(x,1) = 2x$$

Having Analytical Solution as:

$$w(x,y) = 2xy^3 \tag{19}$$

This steady state solution of this problem is displayed in Figure 3..

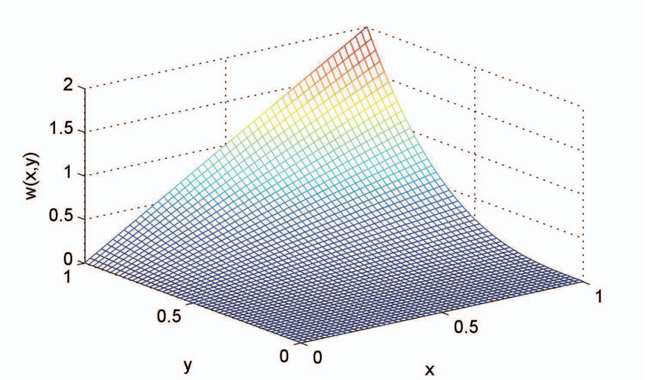

Figure 3. Analytical Solution of Problem P3

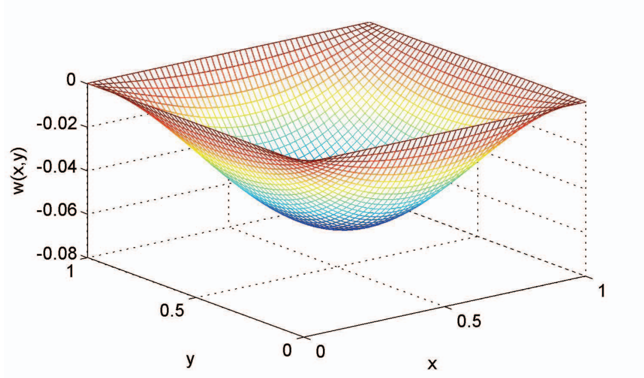

Figure 4. Analytical Solution of Problem P4

D. Sinusoidal Heat Vector (P4)

The Last problem is representative problem of sinusoidal type or having oscillating valued heat vector. This problem is taken from [14]. The problem is:

$$w_{xx} + w_{yy} = \sin(\pi x) \tag{20}$$

With initial conditions as:

$$0 < x, y < 1$$
$$w(0,y) = 0; w(1,y) = 0 \tag{21}$$
$$w(x,0) = 0; w(x,1) = 0$$

Having Analytical Solution as:

$$w(x,y) = \frac{\sin(\pi x)}{\pi^2 \sinh(\pi)} \begin{bmatrix} \sinh(\pi y) + \\ \sinh(\pi(1-y)) \\ -\sinh(\pi) \end{bmatrix} \tag{22}$$

Visual representation of analytical solution is given in Figure 4.

V. Results and Discussion

This section presents the result obtained from the solution of said 4 problems using the Galerkin FEM method described in section III. The results are presented in form of error. The simple error is difference of Numerical and Analytical Solution as presented in Figure 9, Figure 11, Figure 13 and Figure 15. Mean Squared Error or MSE is defiled as:

$$MSE = \frac{1}{N}\sum_{i=1}^{N}\left(w_i^{analytical} - w_i^{numerical}\right)^2 \tag{23}$$

MSE is sometimes called as risk function which corresponds to the mean value of squared error loss may also be called as quadratic loss. The mean absolute relative is or RLE defined as:

$$RLE = \frac{1}{N}\sum_{i=1}^{N}\left|\frac{w_i^{analytical} - w_i^{numerical}}{w_i^{analytical}}\right| \tag{24}$$

The relative error is sometimes also termed as approximation error. All the problems were analyzed on a unit square grid to compare them on same ground using the respective boundary conditions indicated in previous section IV.

The mean squared error of all the problems for steady state solution is presented in Figure 5. The error obtained is analyzed with the increasing number of finite elements count. For all the problems except P2 (which need to be addressed separately) it is evident that initially error decrease somewhat rapidly as the number of finite elements are increased but latterly increase in number of finite elements does not improve accuracy significantly.

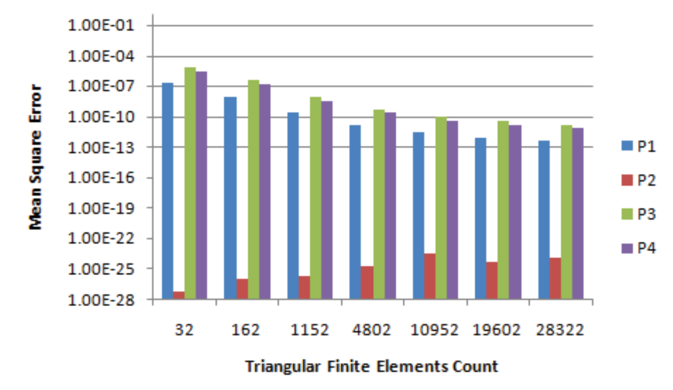

Figure 5. MSE associated with number of FEM elements

Same effect of mean relative error is also observed for accuracy of this method and is presented in Figure 6. Initially error has higher slope with the increase in number of elements indicating decrease in error, lately there is no significant effect is observed on accuracy of solution with increase in number of finite elements.

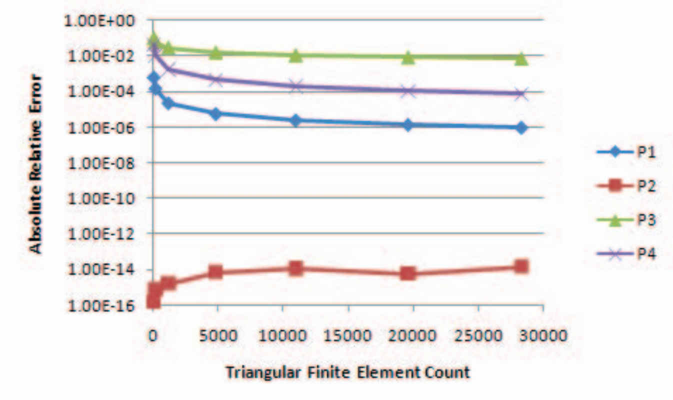

Figure 6. RLE trend with increasing number of FEM elements

The Figure 7 indicates the computational time requirements with the increase in number of FEM elements. It is evident that time requirement grow exponentially as the number of triangular elements are increased. However as indicated in Figure 5 and Figure 6, the increased number of triangular elements does not contribute significantly toward accuracy of the solution. Thus a tradeoff can easily be made of between speed and accuracy of solution.

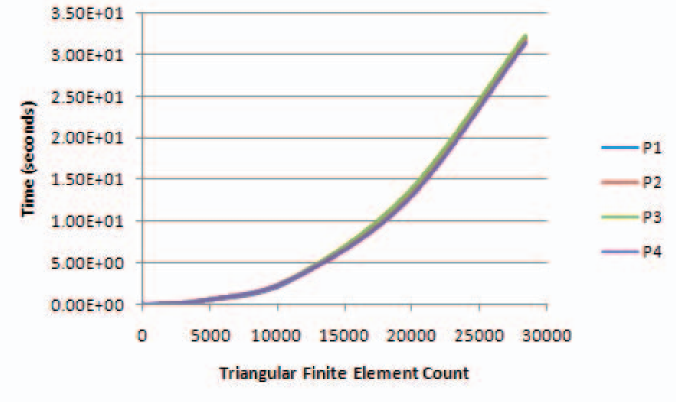

Figure 7. Execution time required with increasing number of FEM elements

Behavior of heat flow vector for P1 is presented in Figure 8 and Figure 9 shows the numerical error resulted from the solution of P1 using Galerkin method. The heat flow vector for this problem is of even type (remains always non negative) and the trend of error observed is somewhat like typical numerical solution. The error is zero at boundaries (it's obvious) and increases as the solution progresses toward middle where this method inhabits maximum error.

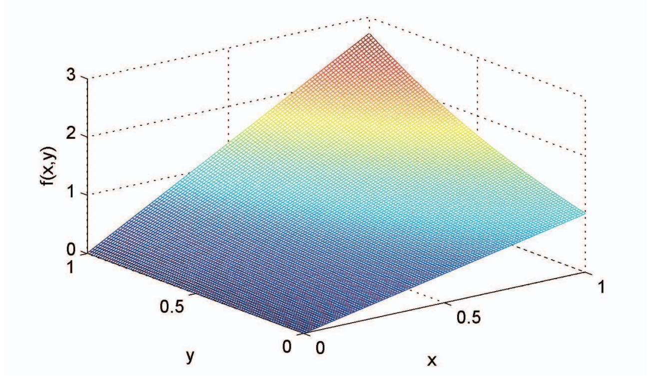

Figure 8. Heat Flow Vector for Problem P1

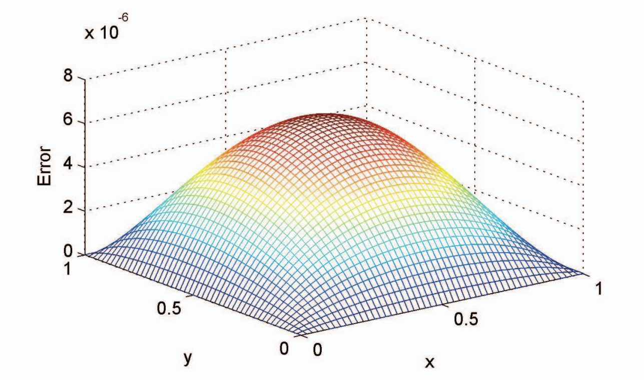

Figure 9. Numerical error resulted for Problem P1

The Figure 10 displays the heat flow vector for P2. From equation (14) we see that the heat flow vector for this problem is zero. It is special case of Poisson equation called as Laplace equation. Figure 11 displays the numerical error resulted from solution of P2. The error resulted for solution of this case is in order of 1.0e-13. This low value of error may be considered as negligible.

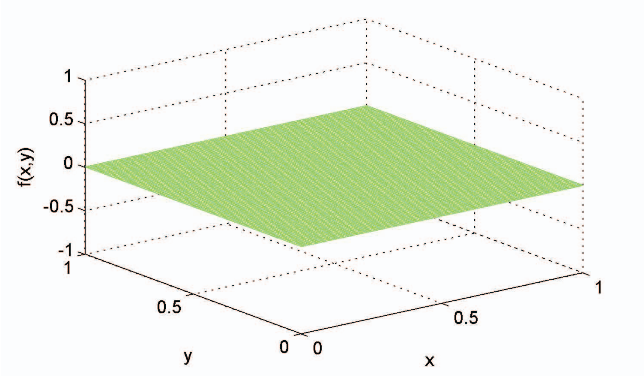

Figure 10. Heat Flow Vector for Problem P1

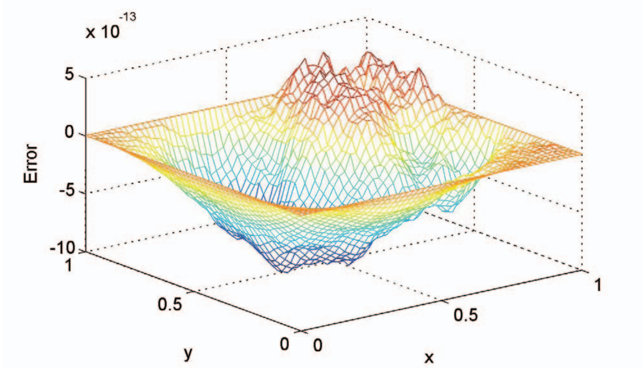

Figure 11. Numerical error resulted for Problem P2

Here in Figure 12 heat flow vector for problem P3 is graphed. The trend is monotonically increasing up to maximum peak of 12. Again the function is of even nature. The next Figure 13 presents the error plot for problem P3. Again the error is non-zero always having same trend as observed for problem P1.

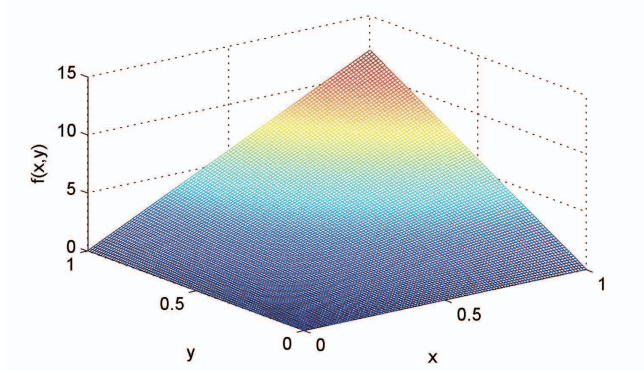

Figure 12. Heat Flow Vector for Problem P3

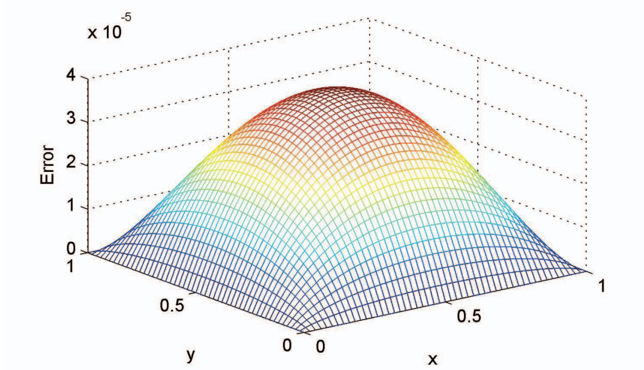

Figure 13. Numerical error resulted for Problem P3

Next Figure 14 shows heat flow vector for problem P4. Due to the even nature of the function it is bind to 0 in y-axis and has sin curves along x-axis. The maximum peak value of this heat flow vector is 1. Numerical error for problem P3 is plotted in Figure 15. The error is always negative which is due to the odd nature of function.

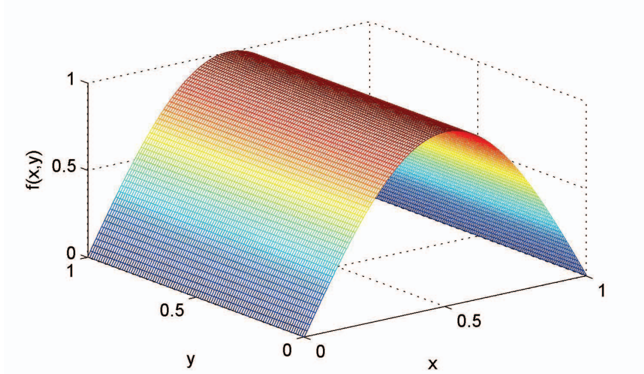

Figure 14. Heat Flow Vector for Problem P4

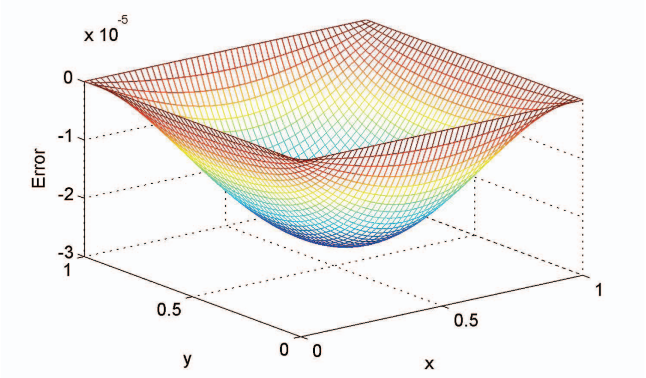

Figure 15. Numerical error resulted for Problem P4

VI. Summary

Closely inspecting all the results and generalizing the results we come to conclusion that error resulted is proportional to the magnitude of the heat flow vector for the FEM method presented in this paper. We also conclude that for even nature heat flow vectors error is always non negative and for odd nature functions the error is negative.

Another important observation is that number of increase in finite elements does not increase the accuracy of solution significantly after crossing some specific number; instead it only increases the computational time exponentially.

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