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Robust observer-based model predictive control of non-uniformly sampled systems

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ABSTRACT

This paper presents a robust model predictive control design procedure for constrained non-uniformly sampled systems subject to bounded unknown disturbances. Unlike the existing results in this area, the proposed design procedure considers arbitrary variations of the sampling period between a lower and an upper bound, and ensures asymptotic stability and performance of the closed-loop control system for all variations of the sampling period, which is a theoretically challenging task. The uncertain optimization problem associated with the constrained model predictive control turns out to be intractable due to infinity many variations of the sampling period. To overcome this challenge, the non-uniformly sampled system is modeled as a linear polytopic uncertain system based on variations in the sampling period. A quasi min–max robust model predictive control technique based on an offline state observer is used for designing an output feedback controller. A case study is given to illustrate the effectiveness of the proposed approach.

1. Introduction

Due to their widespread applications, networked/embedded control systems and event-based control systems are topics of active research [1–3]. For these systems, the uniform sampling of measurements may not be a natural or feasible choice and, hence, neither their periodic operation. This could happen for a number of reasons, including unpredictable sensor delays, packet dropouts, the nature of the network connecting the control system components, and event-triggering mechanism. In addition, due to the embedded-systems approach to their implementation, sampling and updating periods may become time-varying due to processing and monitoring of several tasks. Non-uniformly sampled systems provide a suitable modeling abstraction to study the controlled behavior of these classes of systems under the presence of aforementioned effects [4].

Non-uniformly sampled systems (NUSS) arise when the sampling and input updating periods are varying arbitrarily and find many real world applications, including paper machine systems [5], brushless DC servo systems [6], and networked control systems [7,8]. The control of non-uniformly sampled systems is a challenging task because it is not predictable when the measurements are transmitted to the controller and when the control signals are updated. These systems, therefore, vary inherently in time and so should their controls. Different modeling and analysis approaches for NUSS are available in the literature, which can broadly be classified into the continuous-time approaches and the discrete-time approaches. In the continuous-time approaches, a sampled-data system is modeled as a time delay system, an impulsive system, or a system with a norm-bounded uncertainty. In the discrete-time approaches, the non-uniformly sampled system can be modeled as a polytopic uncertain systems or as a linear fractional transformation. Representative examples for the continuous-time approaches include [9–12] and for the discrete-time [8,13,14]. Also, different control schemes are proposed, such as, state feedback control [15,16], observer-based control [17], dynamic output-feedback control [18,19], $\mathcal{H}_2$ control [20], $\mathcal{H}_\infty$ control [21,22], generalized predictive control [23], and model predictive control [24,25]. It is numerically experienced that discrete-time approaches are less conservative than continuous-time approaches [8,18].

On the other hand, model predictive control (MPC) has become a popular control technique with numerous industrial applications [26–28]. This is primarily due to its inherent capability of handling physical process constraints in an optimized manner [29]. However, due to the presence of inevitable process uncertainties and exogenous perturbations, the performance of a model predictive controller significantly deteriorates [30]. This situation demands the development of robust model predictive control schemes to cope with process uncertainties and disturbance while satisfying the stringent performance requirements

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and physical constraints [31,32]. A survey of existing literature reveals that a number of robust output feedback model predictive control techniques are developed [33–40]. It is pertinent to note that these techniques are developed for systems with uniform sampling. In [23], a generalized predictive control scheme is developed for a special class of non-uniformly sampled systems where the measurements are sampled non-uniformly in a base period and then the whole pattern is repeated in the next base period. In [24], a state-feedback MPC is developed for systems with input saturation. In [29], an observer-based output feedback MPC is developed for systems with missing measurements. The output is sampled at non-uniform intervals but the input is updated at uniform intervals. The two intervals must commensurate for the technique to work. To the authors' best knowledge, the observer-based robust model predictive control of non-uniformly sampled systems with arbitrary sampling periods is not addressed in the existing literature.

In this paper, an observer-based robust model predictive control method is developed to stabilize uncertain sampled-data systems at certified performance where the uncertainty arises from variations of sampling period. The sampling period is allowed to vary arbitrarily between a lower and an upper bound. This scenario is more general than the existing ones [23,24,29]. The non-uniformly sampled system is modeled as a polytopic uncertain system using the approximation technique based on non-uniform variations in the sampling period. A quasi min–max technique with one free control move is used to design an observer-based robust model predictive controller [37]. The control design problem is formulated as an online optimization problem, which is recursively solved at each sampling instant. A challenging task in robust model predictive control is to ensure asymptotic closed-loop stability and recursive feasibility. It is demonstrated that the proposed approach ensures recursive feasibility of the optimization problem and the designed controller successfully stabilizes the closed-loop system for all variations of the sampling period.

The rest of this article is organized as follows. Section 2 describes the non-uniformly sampled feedback control scenario and its mathematical formulation. Section 3 describes the polytopic approximation of the NUSS. Section 4 presents the two-step approach to designing an output feedback controller by first designing an offline observer and then a quasi min–max robust model predictive controller. Section 6 presents a case study to which the proposed procedure has been applied and finally, concluding remarks are provided in Section 7.

2. Problem formulation

Consider a process control scenario in a networked environment as shown in Fig. 1. The communication between the physical process and the controller takes place via a shared communication network. In this configuration, $(y_1(t), \dots, y_m(t))$ are continuous-time signals representing the process variables that can be measured. The non-uniform samplers, as the name suggests, sample the plant outputs at non-uniform sampling intervals and transmit to controller via a communication network. The measurements $y_k = \left(y_{1,k}, \dots, y_{m,k}\right)$ received at the controller side are used by the observer to estimate the process state. The estimated state is then used to compute the desired control sequence, $u_k = \left(u_{1,k}, \dots, u_{p,k}\right)$. The controller is event-driven; as soon as new data is received, the process state is estimated and a new control action is computed. The computed control action is transmitted over the communication network and applied to the plant via zero-order hold devices. Our goal is to design an observer-based model-predictive controller that uses non-uniformly sampled measurements, maintains the stability of the closed-loop system and satisfies performance requirements for all variations of the sampling period.

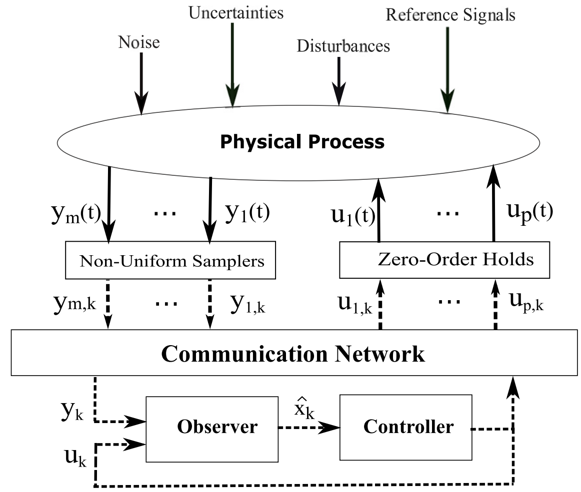

Fig. 1. Non-uniformly sampled output feedback process control.

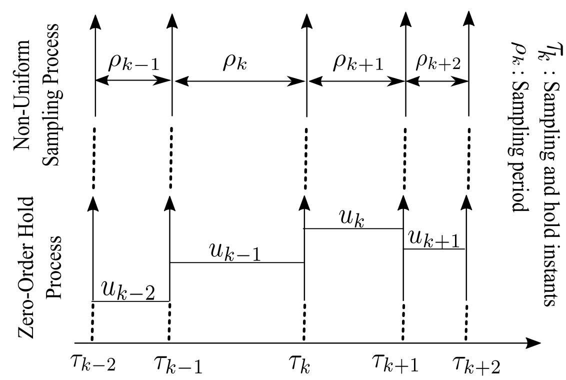

Fig. 2. Non-uniform sampling and hold processes.

2.1. Process model

The dynamics of the process can be described by the following continuous-time state space model

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Dw(t) \\ y(t) &= Cx(t) + Ew(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the process state, $u(t) \in \mathbb{R}^p$ is the control input, $y(t) \in \mathbb{R}^m$ is the measurable process output, and $w(t) \in \mathbb{R}^w$ is the persistent unknown disturbance such that $w(t)^T w(t) \leq 1$. A, B, C, D, and E are matrices of compatible dimensions.

2.2. Non-uniform sampling of the process

Fig. 2 shows a schematic of the non-uniform sampling and hold processes. Assume that there are no packet dropouts and communication delays in the network.

- The measurement vector $y(t) = (y_1(t), \dots, y_m(t))^T$ from the process is sampled by non-uniform samplers when $t = \tau_k$, where $\{\tau_k : k \geq 0\}$ is a set of arbitrary non-uniform sampling instants with properties

$$0 < \rho_l \leq \rho_k = \tau_{k+1} - \tau_k \leq \rho_u < \infty \tag{2a}$$

$$\tau_0 = 0, \quad \lim_{k\to\infty} \tau_k = \infty. \tag{2b}$$

The symbols ρ_l and ρ_u denote the lower and upper limits of variation in the sampling interval, ρ_k. The property (2b) is required to avoid Zeno's phenomenon.

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- The control input vector $u(t) = (u_1(t), \ldots, u_p(t))^T$ to the process is generated using zero-order-hold devices such that

$$u(t) = u(\tau_k) = u_k, \quad t \in [\tau_k, \quad \tau_{k+1}]$$

- Both the zero-order hold devices and samplers are synchronized.

Under these assumptions, a discrete-time model of process in (1) at sampling instants, τ_k, is given as

$$\begin{aligned} x_{k+1} &= A_d(\rho_k)x_k + B_d(\rho_k)u_k + D_d(\rho_k)w_k \\ y_k &= Cx_k + Ew_k \end{aligned} \tag{3}$$

where $x_k := x(\tau_k)$, $w_k := w(\tau_k)$, $y_k := y(\tau_k)$, and

$$A_d(\rho_k) := e^{\rho_k A}, \quad B_d(\rho_k) := \int_0^{\rho_k} e^{(\rho_k - \nu)A} d\nu B$$

$$D_d(\rho_k) := \int_0^{\rho_k} e^{(\rho_k - \nu)A} d\nu D$$

2.3. Observer-based model predictive control

Our aim is to design an observer-based model predictive controller for the process in Fig. 1. We adopt the following state observer to estimate the state of the system.

$$\hat{x}_{k+1} = A_d(\rho_k)\hat{x}_k + B_d(\rho_k)u_k + L_o(y_k - C\hat{x}_k) \tag{4}$$

where $\hat{x}_k \in \mathbb{R}^n$ is the estimated state and L_o is the observer gain matrix to be designed. Based on the estimated state, $\hat{x}_k$, the control action is computed as state feedback having the following form

$$u_k = F_k\hat{x}_k \tag{5}$$

where F_k is the state feedback gain matrix to be designed. Defining the estimation error $e_k = x_k - \hat{x}_k$ and using (3) and (4), the dynamics of estimation error can be written as

$$e_{k+1} = [A_d(\rho_k) - L_oC]\, e_k + [D_d(\rho_k) - L_oE]\, w_k \tag{6}$$

The prediction model can then be written as

$$\begin{bmatrix} \hat{x}_{k+1} \\ e_{k+1} \end{bmatrix} = \begin{bmatrix} \xi(\rho_k) & L_oC \\ 0 & \zeta(\rho_k) \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ e_k \end{bmatrix} + \begin{bmatrix} L_oD_d(\rho_k) \\ \varphi(\rho_k) \end{bmatrix} w_k \tag{7}$$

where $\xi(\rho_k) = A_d(\rho_k) + B_d(\rho_k)F_k$, $\zeta(\rho_k) =: A_d(\rho_k) - L_oC$ and $\varphi(\rho_k) := D_d(\rho_k) - L_oE$.

Let $\hat{x}_{i|k}$ be the value of $\hat{x}$ at time instant $k+i$ predicted at time instant k and $u_{i|k}$ be the value of u at time instant $k+i$ computed at time instant k. Define the performance objective to be minimized as

$$J_\infty(k) = \sum_{i=0}^{\infty} \begin{bmatrix} \hat{x}_{i|k} \\ u_{i|k} \end{bmatrix}^T \begin{bmatrix} \mathcal{W}_x & 0 \\ 0 & \mathcal{W}_u \end{bmatrix} \begin{bmatrix} \hat{x}_{i|k} \\ u_{i|k} \end{bmatrix} \tag{8}$$

where $\mathcal{W}_x > 0$ and $\mathcal{W}_u > 0$ are suitable weighting matrices.

2.4. Constraints

The controlled system is required to satisfy the following constraint on the plant inputs.

$$u_{i|k} \in \left\{u| -\bar{u}^j \le u^j \le \bar{u}^j,\ j = 1, \ldots, p\right\}, \quad \forall i \ge 0 \tag{9}$$

where $\bar{u}^j$ denotes the given peak bound on the jth input.

The observer-based model predictive control problem for non-uniformly sampled processes can be stated as follows:

Problem 1. Given the non-uniformly sampled process in (3) subjected to constraint (9), design the observer-based controller in (4) and (5) such that the output feedback closed-loop system is exponentially stable for all variations of the sampling period in the presence of unknown disturbances and initial conditions, and the performance index in (8) is minimized.

Problem 1 can mathematically be expressed as the following uncertain robust optimization problem

$$\min_{u_{i|k}} \max_{[A_d(\rho_{k+i}), B_d(\rho_{k+i}), D_d(\rho_{k+i})]\ \forall \rho_{k+i} \in [\rho_l,\ \rho_u],\ i \ge 0} J_\infty(k) \tag{10}$$

subject to (3)–(5) and (7)–(9).

The optimization problem (10) is much harder to solve because the parameters of the closed-loop are dependent on sampling period ρ_k, which can take arbitrary values in the interval $[\rho_l,\ \rho_u]$. That means the performance index needs to be maximized and minimized, and constraints are required to be satisfied for infinity-many values of the plant parameters, which is computationally impossible [41].

3. Polytopic approximation

As pointed out in the previous section, Problem 1 is computationally intractable. One way to make the problem tractable is to approximate the non-uniformly sampled system (3) with a polytopic uncertain system based on variations in the sampling period. There are different methods for the polytopic approximation of an uncertain systems [42]. We use the Cayley–Hamilton method. By using the Cayley–Hamilton theorem, the matrix $A_d(\rho_k)$ can be written as

$$A_d(\rho_k) = e^{\rho_k A} = \mu_0(\rho_k)I + \mu_1(\rho_k)A + \cdots + \mu_{n-1}(\rho_k)A^{n-1}$$

where $\mu_i(\rho_k)$, $i = 0, \ldots, n-1$ are the sampling period dependent coefficients of the characteristic equation. Since $\rho_k \in [\rho_l,\ \rho_u]$, therefore $\mu_i(\rho_l) \le \mu_i(\rho_k) \le \mu_i(\rho_u)$. Let $\lambda_i, i = 1, \ldots, n$ be the eigenvalues of the matrix A, then the coefficients $\mu_i(\rho_k)$ can be computed as follows

$$\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \ldots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} \mu_0(\rho_k) \\ \mu_1(\rho_k) \\ \vdots \\ \mu_{n-1}(\rho_k) \end{bmatrix} = \begin{bmatrix} e^{\rho_k\lambda_1} \\ e^{\rho_k\lambda_2} \\ \vdots \\ e^{\rho_k\lambda_n} \end{bmatrix}$$

Define boundary matrices of $A_d(\rho_k)$ as

$$\begin{aligned} A_d^1 &= \mu_0(\rho_l)I + \mu_1(\rho_l)A + \ldots + \mu_{n-1}(\rho_l)A^{n-1} \\ A_d^2 &= \mu_0(\rho_l)I + \mu_1(\rho_l)A + \ldots + \mu_{n-1}(\rho_u)A^{n-1} \\ &\vdots \\ A_d^{n_v} &= \mu_0(\rho_u)I + \mu_1(\rho_u)A + \ldots + \mu_{n-1}(\rho_u)A^{n-1}, \end{aligned}$$

The matrix $A_d(\rho_k)$ can be then embedded to the following polytopic uncertainty set

$$A_d(\rho_k) \in \left\{ \sum_{i=1}^{n_v} \delta^i(\rho_k)A_d^i,\ \delta^i(\rho_k) \ge 0,\ \sum_{i=1}^{n_v} \delta^i(\rho_k) = 1 \right\}$$

where n_v is the number of vertices of the polytope and $\delta^i(\rho_k)$ are the convex coefficients. A similar procedure can be followed to embed the matrices $B_d(\rho_k)$ and $D_d(\rho_k)$ into polytopic uncertain sets. Let the boundary matrices of $B_d(\rho_k)$ be $B_d^1, \ldots .B_d^{n_v}$ and $D_d(\rho_k)$ be $D_d^1, \ldots, D_d^{n_v}$, the uncertain matrices in (3) can then be embedded into the polytope, $\begin{bmatrix} A_d(\rho_k) & B_d(\rho_k) & D_d(\rho_k) \end{bmatrix} \in \Omega$, where

$$\Omega = \left\{ \sum_{i=1}^{n_v}\sum_{j=1}^{n_v} \delta^i(\rho_k)\omega^j(\rho_k) \begin{bmatrix} A_d^i & B_d^j & D_d^j \end{bmatrix} \right\},$$

$$\delta^i(\rho_k) \geq 0,\ \omega^j(\rho_k) \geq 0, \quad \sum_{i=1}^{n_v}\sum_{j=1}^{n_v} \delta^i(\rho_k)\omega^j(\rho_k) = 1$$

In this paper, we assume that the current sampling period ρ_k is measurable and hence the system matrices $[A_d(\rho_k),\ B_d(\rho_k),\ D_d(\rho_k)]$ are known exactly at current sampling period ρ_k. However, the future ones $[A_d(\rho_{k+i}),\ B_d(\rho_{k+i}),\ D_d(\rho_{k+i})]$, $\forall i \geq 1$, are not known and vary within the predefined polytope Ω. The optimization problem (10) can then be put into a tractable form as

$$\min_{u_{i|k}} \max_{[A_d(\rho_{i|k}), B_d(\rho_{i|k}), D_d(\rho_{i|k})] \in \Omega,\ i \geq 0} J_\infty(k) \tag{11}$$

subject to (3)–(5) and (7)–(9).

This is the so-called min–max optimization problem with polytopic uncertainty description. Different techniques exist to solve this problem such as [36–40]. We follow the approach similar to [39], which involves designing an offline estimator and an online controller via a quasi min–max optimization technique.

4. Observer-based quasi min–max robust MPC design

We follow the quasi min–max approach to solve the infinite horizon optimization problem (11). The performance index $J_\infty(k)$ in (8) can be split into two parts as

$$J_\infty(k) = J_0^1(k) + J_1^\infty(k)$$

where

$$J_0^1(k) = \left\|\hat{x}_{0|k}\right\|^2_{\mathcal{W}_x} + \left\|u_{0|k}\right\|^2_{\mathcal{W}_u}$$

$$J_1^\infty(k) = \sum_{i=1}^{\infty}\left\{\left\|\hat{x}_{i|k}\right\|^2_{\mathcal{W}_x} + \left\|u_{i|k}\right\|^2_{\mathcal{W}_u}\right\}$$

where J_0^1 is the first stage cost function, and J_1^∞ is the remaining cost function. The control vector that minimizes J_0^1 and J_1^∞ is given as

$$U_0^\infty(k) = \begin{cases} u_{0|k}, & i = 0 \\ F_{i|k}\hat{x}_{i|k}, & i \geq 1 \end{cases}$$

The associated prediction model can be written as

$$\begin{bmatrix} \hat{x}_{i+1|k} \\ e_{i+1|k} \end{bmatrix} = \begin{bmatrix} \xi_{i|k} & L_o C \\ 0 & \zeta_{i|k} \end{bmatrix} \begin{bmatrix} \hat{x}_{i|k} \\ e_{i|k} \end{bmatrix} + \begin{bmatrix} L_o E \\ \varphi_{i|k} \end{bmatrix} w_{k+i}, \quad i \geq 1 \tag{12}$$

where

$$\hat{x}_{k+i|k} = \hat{x}_{i|k},\ \ e_{k+i|k} = e_{i|k},\ \ \xi_{i|k} = A_d(\rho_{i|k}) + B_d(\rho_{i|k})F_k,$$
$$\zeta_{i|k} = A_d(\rho_{i|k}) - L_o C,\ \ \varphi_{i|k} = D_d(\rho_{i|k}) - L_o E$$

The first control move, $u_{0|k}$, is computed as a free variable while the remaining elements are computed as a state-feedback to minimize the upper-bound on worst-case value of the cost function J_1^∞. Fig. 3 elaborates the working of quasi-min–max algorithm.

4.1. Offline robust observer design

For the error system in (6), define the Lyapunov function

$$\Phi_{e,k} = e_k^T W_e e_k = \|e_k\|^2_{W_e}, \quad W_e > 0.$$

According to [43], the error system will be exponentially bounded in the presence of unknown initial conditions, e_0, disturbances, w_k, and for all $[A_d(\rho_k), B_d(\rho_k), D_d(\rho_k)] \in \Omega$, if

$$\Phi_{e,k} \geq 1 \implies \Phi_{e,k+1} \leq \Phi_{e,k} \tag{13}$$

hold. The observer gain L_o in (4) can be designed to satisfy the quadratic boundedness condition in (13), by the following lemma.

Lemma 1. *Given the error system in (6) and scalars $s^{i,j}_{\rho_k} \geq 0$, if there exist matrices $W_e > 0$, Z_e satisfying*

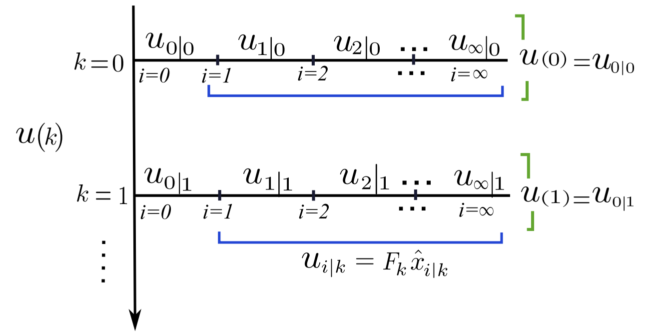

Fig. 3. Quasi min–max algorithm.

$$\max_{W_e > 0,\ Z_e} \ \text{trace}\,(W_e)$$

$$\begin{bmatrix} (1 - s^{i,j}_{\rho_k})W_e & \star & \star \\ 0 & s^{i,j}_{\rho_k} I & \star \\ W_e A_d^i - Z_e C & W_e D_d^j - Z_e E & W_e \end{bmatrix} \geq 0 \tag{14}$$

for $i = 1, 2, \ldots, n_v$, $j = 1, 2, \ldots, n_v$, then the condition (13) is satisfied. This results in convergence of future trajectories of error e_k to the invariant ellipsoid $\mathcal{E}_{W_e} = \left\{e_k |\ \|e_k\|^2_{W_e} \leq 1\right\}$ and will stay in it thereafter. The observer gain is given by

$$L_o = (W_e)^{-1} Z_e$$

Proof. See Appendix A.

Remark 1. In (14), $s^{i,j}_{\rho_k}$ are vertices of the sampling period-dependent decay rate $s(\rho_k)$. Their values can be line-searched over the interval (0, 1). To reduce the computational burden, one can also fix or equalize the values of $s^{i,j}_{\rho_k}$.

4.2. Online control design

Before we state the main theorem for the controller design, we make the following assumption.

Assumption 1. $\|e_1\|^2_{W_e} \leq \eta_1$, where η_1 is a user-specified constant. At each sampling instant $k \geq 1$, the bound of e_{k+1} is refreshed as

$$\|e_{k+1}\|^2_{W_e} \leq \eta_{k+1}. \tag{15}$$

Theorem 1. *Given the process in (1) with its non-uniformly sampled model in (3) subjected to input constraints (9), the observer-based controller given by (4) and (5) that will robustly stabilize the closed-loop system and minimize the performance index in (8) will be given by the online solution of the following semidefinite optimization problem at each non-uniformly sampled time instant, τ_k*

$$\min_{u_{0|k}, Q_k, Z_k, X_k, \sigma_k^{i,j}, \beta_k, \gamma_k} \gamma_k$$

Subject to

$$\begin{bmatrix} (1-\sigma_k^{i,j})Q_k & \star & \star & \star & \star & \star \\ 0 & m_{22} & \star & \star & \star & \star \\ 0 & m_{32} & m_{33} & \star & \star & \star \\ A_d^i Q_k + B_d^j Z_k & L_o C & L_o E & Q_k & \star & \star \\ \mathcal{W}_x^{\frac{1}{2}} Q_k & 0 & 0 & 0 & \gamma_k I & \star \\ \mathcal{W}_u^{\frac{1}{2}} Z_k & 0 & 0 & 0 & 0 & \gamma_k I \end{bmatrix} \geq 0 \tag{16}$$
$$i \in \{1, \ldots, n_v\}, \quad j \in \{1, \ldots, n_v\}$$

$$m_{22} = \frac{\beta_k}{\eta_{k+1}}\left[\left(1-\sigma_k^{i,j}\right)W_e - (\zeta^i)^T W_e \zeta^i\right],$$
$$m_{32} = -\frac{\beta_k}{\eta_{k+1}}(\varphi^j)^T W_e \zeta^i,$$
$$m_{33} = \sigma_k^{i,j} I - \frac{\beta_k}{\eta_{k+1}}(\varphi^j)^T W_e \varphi^j,$$
$$\zeta^i = A_d^i - L_o C, \quad \varphi^j = D_d^j - L_o E$$

$$\begin{bmatrix} 1-\beta_k & \star & \star & \star \\ \Xi_{21} & Q_k & \star & \star \\ \mathcal{W}_x^{\frac{1}{2}}\hat{x}_k & 0 & \gamma_k I & \star \\ \mathcal{W}_u^{\frac{1}{2}} u_{0|k} & 0 & 0 & \gamma_k I \end{bmatrix} \geq 0, \tag{17}$$

$$\Xi_{21} = A_d(\rho_k)\hat{x}_k + B_d(\rho_k)u_{0|k} + L_o(y_k - C\hat{x}_k).$$

$$\eta_{k+1} = \min\left\{\frac{\tilde{\eta}_{k+1}}{\alpha_{k-1}}, \eta'_{k+1}, \eta''_{k+1}\right\}, \tag{18}$$

$$\tilde{\eta}_{k+1} = \gamma_{k-1} - (1-\sigma_{1|k-1})(\left\|\hat{x}_{k-1}\right\|^2_{\mathcal{W}_x} - \left\|\hat{x}_k\right\|^2_{\mathcal{W}_x + F_{k-1}^T \mathcal{W}_u F_{k-1}}$$
$$+ \ \|u_{k-1}\|^2_{\mathcal{W}_u}) - \left\|\tilde{x}_{1|k}\right\|^2_{P_k},$$
$$\tilde{x}_{1|k} = (A_d(\rho_k) + B_d(\rho_k)F_{k-1})\hat{x}_k + L_o(y_k - C\hat{x}_k)$$
$$\eta'_{k+1} = 1 + (1 - s(\rho_k))(\eta_k - 1)$$
$$\eta''_{k+1} = (1 - \sigma_{1|k-1} + \frac{\sigma_{1|k-1}}{\beta_{k-1}})\eta_k$$

$$\begin{bmatrix} X_k & \star \\ Z_k^T & Q_k \end{bmatrix} \geq 0, \quad X_{jj} \leq (\bar{u}^j)^2, \quad j \in \{1, \ldots, p\}, \tag{19}$$

where $\alpha_k = \frac{\beta_k\gamma_k}{\eta_{k+1}}$, X_k^{jj} is the jth diagonal element of X_k.

Proof. To solve the min–max robust optimization problem (11), we first derive an upper bound on $J_1^\infty(k)$. For that, let us define a quadratic Lyapunov function

$$\Phi_{i|k} = \left\|\hat{x}_{i|k}\right\|^2_{P_k} + \left\|e_{i|k}\right\|^2_{W_{e,k}}, \quad i \geq 1 \tag{20}$$

where $P_k > 0$ and $W_{e,k} = \alpha_k W_e > 0$. Let us suppose $\Phi_{i|k}$ satisfies the following robust stability condition at each sampling instant τ_k

$$\Phi_{i+1|k} - \Phi_{i|k} \leq -\left(\left\|\hat{x}_{i|k}\right\|^2_{\mathcal{W}_x} + \left\|u_{i|k}\right\|^2_{\mathcal{W}_u}\right) \quad \forall i \geq 1 \tag{21}$$

To guarantee asymptotic stability, we have to ensure that $\hat{x}_{\infty|k} = 0$ so that $\Phi_{\infty|k} = \Phi(\hat{x}_{\infty|k}) = 0$. Now summing (21) from $i = 1$ to ∞, we get

$$-\Phi_{1|k} \leq -J_1^\infty(k)$$

Thus

$$\max_{[A_d(\rho_{k+i}), B_d(\rho_{k+i}), D_d(\rho_{k+i})] \in \Omega, \ i \geq 1} J_1^\infty(k) \leq \Phi_{1|k}$$

The value of the performance index $J_\infty(k)$ will be bounded by

$$J_\infty(k) \leq \left\|\hat{x}_{0|k}\right\|^2_{\mathcal{W}_x} + \left\|u_{0|k}\right\|^2_{\mathcal{W}_u} + \Phi_{1|k}$$

Let γ_k be an upper bound as follows

$$\left\|\hat{x}_{0|k}\right\|^2_{\mathcal{W}_x} + \left\|u_{0|k}\right\|^2_{\mathcal{W}_u} + \Phi_{1|k} \leq \gamma_k. \tag{22}$$

The state of the closed-loop system (7) will be exponentially bounded, if for the Lyapunov function (20) satisfies

$$\Phi_{i|k} \geq \gamma_k \implies (21). \tag{23}$$

Since w_k is uncertain, persistent and satisfies $\|w_k\|^2 \leq 1$, $\Phi_{i|k} \geq \gamma_k$ is equivalent to $\Phi_{i|k} \geq \gamma_k \|w_{k+i}\|^2$ for all $\|w_{k+i}\| \leq 1$. By invoking the S-procedure, (23) is satisfied if there exist scalars $\sigma_{i|k} > 0$ such that

$$\begin{aligned} &\left\|\tilde{x}_{i|k}\right\|^2_{P_k} + \left\|e_{i|k}\right\|^2_{W_{e,k}} - \left\|\zeta_{i|k}e_{i|k} + \zeta_{i|k}w_{k+i}\right\|^2_{W_{e,k}} \\ &- \ \left\|\zeta_{i|k}\hat{x}_{i|k} + L_o C e_{i|k} + L_o E w_{k+i}\right\|^2_{P_k} - \left\|\hat{x}_{i|k}\right\|^2_{\mathcal{W}_x + F_k^T \mathcal{W}_u F_k} \\ &- \ \sigma_{i|k}(\left\|\tilde{x}_{i|k}\right\|^2_{P_k} + \left\|e_{i|k}\right\|^2_{W_{e,k}} - \gamma_k \|w_{k+i}\|^2) \geq 0, \ \exists \sigma_{i|k} > 0 \end{aligned} \tag{24}$$

By removing $\{\hat{x}_{i|k}, e_{i|k}, w_{k+i}\}$ and applying Schur complement followed by congruence transformations and substitutions such as $Q_k = \gamma_k {P_k}^{-1}$ and $Z_k = F_k Q_k$, we can conclude that (24) holds if and only if

$$\begin{bmatrix} p_{11} & \star & \star & \star & \star & \star & \star \\ 0 & p_{22} & \star & \star & \star & \star & \star \\ 0 & 0 & \sigma_{i|k} I & \star & \star & \star & \star \\ p_{41} & L_o C & L_o E & Q_k & \star & \star & \star \\ 0 & \zeta_{i|k} & \zeta_{i|k} & 0 & \gamma_k W_{e,k}^{-1} & \star & \star \\ \mathcal{W}_x^{\frac{1}{2}} Q_k & 0 & 0 & 0 & 0 & \gamma_k I & \star \\ \mathcal{W}_u^{\frac{1}{2}} Z_k & 0 & 0 & 0 & 0 & 0 & \gamma_k I \end{bmatrix} \geq 0 \tag{25}$$

$$p_{11} = (1 - \sigma_{i|k})Q_k, \quad p_{22} = (1 - \sigma_{i|k}){\gamma_k}^{-1} W_{e,k}$$
$$p_{41} = A_d(\rho_{i|k})Q_k + B_d(\rho_{i|k})Z_k$$

Taking the scalars $\sigma_{i|k}$ to be sampling period dependent as $\sigma_{i|k} = \sum_{i=1}^{n_v}\sum_{j=1}^{n_v}\delta^i(\rho_k)\omega^j(\rho_k)\sigma_k^{i,j}$ and noting that (25) is affine in $[A_d|B_d|D_d|\sigma](\rho_{i|k})$, it is equivalent to replace $[A_d|B_d|D_d|\sigma](\rho_{i|k})$ by $[A_d^i|B_d^j|D_d^j|\sigma^{i,j}]$ for all $i \in \{1, \ldots, n_v\}, j \in \{1, \ldots, n_v\}$. In addition, letting $\alpha_k = \frac{\beta_k\gamma_k}{\eta_{k+1}}$ (i.e., $W_{e,k} = \frac{\beta_k\gamma_k}{\eta_{k+1}}W_e$) and applying Schur complement on (25), we can obtained (16). Now by substituting (20) for i = 1 into (22), we obtain

$$\left\|\hat{x}_{0|k}\right\|^2_{\mathcal{W}_x} + \left\|u_{0|k}\right\|^2_{\mathcal{W}_u} + \left\|\hat{x}_{k+1}\right\|^2_{P_k} + \|e_{k+1}\|^2_{W_{e,k}} \leq \gamma_k \tag{26}$$

Since (15) and $W_{e,k} = \frac{\beta_k\gamma_k}{\eta_{k+1}}W_e$, (26) is guaranteed by

$$\left\|\hat{x}_{0|k}\right\|^2_{\mathcal{W}_x} + \left\|u_{0|k}\right\|^2_{\mathcal{W}_u} + \left\|\hat{x}_{k+1}\right\|^2_{P_k} \leq (1 - \beta_k)\gamma_k \tag{27}$$

Now by substituting $P_k := \gamma_k Q_k^{-1}$, multiplying γ_k^{-1} followed by application of Schur complement, on (27), results into (17).

Now we prove that, if at each sampling instant $\tau_k, k \geq 1$, the inequalities (16) and (17) hold, then Assumption 1 can be guaranteed for all $k \geq 1$. where (18) governs the updation of estimation error bound (EEB) η_{k+1}. For $i = 1$, the constraint expressed in (17), which satisfies (22), results into

$$\Phi_{2|k} \leq (1 - \sigma_{1|k})\Phi_{1|k} - \left\|\hat{x}_{k+1}\right\|^2_{\mathcal{W}_x + F_k^T \mathcal{W}_u F_k} + \sigma_{1|k}\gamma_k \|w_{k+1}\|^2 \tag{28}$$

Applying {(28), (22), (20)}, and $\|w_{k+1}\| \leq 1$, yields

$$\begin{aligned} \left\|e_{2|k}\right\|^2_{W_{e,k}} \leq \gamma_k - (1 - \sigma_{1|k})(\left\|\hat{x}_k\right\|^2_{\mathcal{W}_x} + \left\|u_{0|k}\right\|^2_{\mathcal{W}_u}) \\ - \ \left\|\hat{x}_{k+1}\right\|^2_{\mathcal{W}_x + F_k^T \mathcal{W}_u F_k} - \left\|\hat{x}_{2|k}\right\|^2_{P_k} \end{aligned} \tag{29}$$

For predicting $\{\hat{x}_{2|k}, e_{2|k}\}$ (referring to (12) for $i = 1$), It should take all promising values of $\delta_{1|k}^i \geq 0$ satisfying $\sum_{i=1}^{n_v}\delta_{1|k}^i = 1$, i.e. $\{\hat{x}_{2|k}, e_{2|k}\}$ into consideration in (29) that are not known in $\delta_{1|k}^i$. The signal $\{\hat{x}_{1|k+1}, e_{k+2}\}$, being deterministic in δ_{k+1}^i, are always included in the possible values of $\{\hat{x}_{2|k}, e_{2|k}\}$. Therefore, by replacing $\{\hat{x}_{2|k}, e_{2|k}\}$ in (29) by $\{\hat{x}_{1|k+1}, e_{k+2}\}$, results into

$$\|e_{k+2}\|^2_{W_{e,k}} \leq \tilde{\eta}_{k+2} \tag{30}$$

Now by using Remark 1, we can also obtain

$$\|e_{k+2}\|^2_{W_e} \leq \eta'_{k+2} \tag{31}$$

Furthermore, consider the following block in (25):

$$\begin{bmatrix}(1-\sigma_{i|k})\gamma_k^{-1}W_{e,k} & \star & \star \\ 0 & \sigma_{i|k}I & \star \\ \zeta_{i|k} & \zeta_{i|k} & \gamma_k W_{e,k}^{-1}\end{bmatrix} \geq 0 \tag{32}$$

Analogous to Remark 1, (32) implies that $\|e_{k+2}\|^2_{\gamma_k^{-1}W_{e,k}} \leq 1+(1-\sigma_{1|k})(\|e_{k+1}\|^2_{\gamma_k^{-1}W_{e,k}} - 1)$, which leads to

$$\|e_{k+2}\|^2_{W_e} \leq \eta''_{k+2} \tag{33}$$

Applying {(30), (31), (33)} and $W_{e,k} = \alpha_k W_e$, $\forall k \geq 0$, it is concluded that estimation error bound $\eta_{k+1} \forall k \geq 1$ can be updated according to (18), so that Assumption 1 is satisfied.

To prove (19), consider the peak bounds

$$|u^j_{i|k}| < \bar{u}^j, \quad j = 1, \ldots, p, \quad i \geq 1 \tag{34}$$

It holds that [44]

$$\begin{aligned}\max_{i\,\geq\,1}\ |u^j_{i|k}|^2 &= \max_{i\,\geq\,1}\ |Z_k Q_k^{-1} \hat{x}^j_{i|k}|^2 \\ &\leq \max_{z\,\in\,\mathcal{E}_{inv}} |Z_k Q_k^{-1} z^j|^2 \\ &\leq \left\| \left(Z_k Q_k^{-1}\right)_j \right\|_2^2 \\ &= \left(Z_k Q_k^{-1} Z_k^T\right)_{jj}\end{aligned} \tag{35}$$

Thus, the existence of a symmetric matrix X_k such that

$$\begin{bmatrix} X_k & Z_k \\ Z_k^T & Q_k \end{bmatrix} \geq 0, \text{ with } X_{jj} \leq (\bar{u}^j)^2$$

guarantees that $|u^j_{i|k}| < \bar{u}^j, \quad j = 1, \ldots, p, \quad i \geq 1$. This completes the proof.

Next, we prove that the optimization problem of Theorem 1 remains feasible at all sampling instants.

Lemma 2. *Assume that the optimization problem in Theorem 1 has a solution at the instant k. Then, by taking $P_k = \gamma_k Q_k^{-1}$ and the future feedback control as $u_{i|k} = F_k \hat{x}_{i|k}$, it follows that*

$$\max_{i \geq 1} \quad (\|\hat{x}_{i|k}\|^2_{P_k} + \|e_{i|k}\|^2_{W_{e,k}}) \leq \gamma_k \tag{36}$$

i.e., $\mathcal{E}_{inv} = \{[\hat{x}^T,\ e^T] \in \mathbb{R}^{2n} |\ \|\hat{x}_{i|k}\|^2_{P_k} + \|e_{i|k}\|^2_{W_{e,k}} \leq \gamma_k\}$ is an invariant ellipsoid of the future augmented state predicted by (12).

Proof. Since the optimization problem in Theorem 1 is feasible. The condition (16), which guarantees (24), means that

$$\Phi_{i+1|k} \leq (1-\sigma_{i|k})\Phi_{i|k} - (\|\hat{x}_{i|k}\|^2_{\mathbb{W}_x} + \|u_{i|k}\|^2_{\mathbb{W}_u}) + \sigma_{i|k}\gamma_k \tag{37}$$

$\forall\ i \geq 1$ and $\|w_{k+i}\| \leq 1$. Moreover, (17) guarantees (22). Based on (22), by applying (37) recursively (for $i = 1, 2, \ldots$), it is shown that

$$\hat{x}^T_{i+1|k} P_k \hat{x}_{i+1|k} + e^T_{i+1|k} W_{e,k} e_{i+1|k} \leq \hat{x}^T_{i|k} P_k \hat{x}_{i|k} + e^T_{i|k} W_{e,k} e_{i|k}$$

and thus $\mathcal{E}_{inv} := \{[\hat{x}^T,\ e^T] \in \mathbb{R}^{2n} |\ \|\hat{x}_{i|k}\|^2_{P_k} + \|e_{i|k}\|^2_{W_{e,k}} \leq \gamma_k\}$ is an invariant ellipsoid of the future augmented state predicted by (12).

The procedure to design and implement a robust observer-based model predictive controller for non-uniformly sampled systems can be summarized as follows:

Procedure 1. *Proposed Robust Observer-Based MPC Procedure*

1. *Solve the optimization problem given in Lemma 1 with an additional constraint $S_e \geq cW_e$ to get observer gain L_o. Here, c tunes the size of $\mathcal{E}_{W_e}$. Larger the trace(W_e), smaller the size of $\mathcal{E}_{W_e}$.*
2. *With fixed $\{Z_e, W_e\}$, maximize $s^{i,j}_{\rho_k}$ satisfying (14). Larger the $s^{i,j}_{\rho_k}$, the tighter the estimation error bound η_k.*
3. *Choose a minimum η_0 satisfying $\eta_0 S_e \geq W_e$. Then, $e_0 \in \mathcal{E}_{S_e}$ ensures that $\|e_0\|^2_{W_e} \leq \eta_0$. Choose $\eta_1 = 1+(1-s(\rho_0))(\eta_0-1)$ since $\|e_1\|^2_{W_e} \leq 1+(1-s(\rho_0))(\|e_0\|^2_{W_e}-1)$.*
4. *Initialize k. Measure the current sampling period ρ_k.*
5. *Solve the optimization problem in Theorem 1 based on $\hat{x}_k$ and ρ_k to calculate*
$$u_{i|k} = \begin{cases} u_{0|k} & \text{if } i = 0, \\ F_k \hat{x}_{i|k} & \text{if } i \geq 1. \end{cases} \tag{38}$$
6. *Read the sensor measurements y_k.*
7. *Implement the first control move $u_{0|k}$ to estimate the process state based on the measurement y_k, and ρ_k.*
8. *Set $k = k+1$, wait until next sampling instant and return to step 4.*

The optimization problem of Theorem 1 will be solved at every sampling instant $k \geq 0$. Then, according to receding horizon principle, only the first control move $u_{0|k}$ is implemented to the plant. The optimization problem incorporates the prominent features of a synthesis approach of MPC like robust stability and feasibility, etc.

5. Recursive feasibility and robust stability

Lemma 3. *For the system under consideration subjected to input constraint (9), the optimization problem in Theorem 1 is solved at every sampling instant k whereas the EEB η_{k+1} $\forall$ $k \geq 0$ is updated according to Assumption 1 and (18). Let us assume that the optimization problem of Theorem 1 is feasible for sampling instant k, then*

1. *it will also be feasible for sampling instant $k+1$*
2. *the closed-loop dynamics are robustly stable when $k \rightarrow \infty$,*

Proof. See Appendix B.

Remark 2. Therefore, γ_k, which is the bound of $\|\hat{x}_{0|k}\|^2_{\mathcal{W}_x} + \|u_{0|k}\|^2_{\mathcal{W}_u} + \Phi_{1|k}$, will be monotonically decreasing. For sufficiently large sampling instant k, γ_k will be sufficiently small that ensures the convergence of $\{\hat{x}_k, u_k, \hat{x}_{k+1}, e_{k+1}\}$ in the vicinity of the zero equilibrium point and stay in this vicinity thereafter. For faster convergence of trajectories of estimation error e_k and estimated state $\hat{x}_k$ to equilibrium, $s(\rho_k)$ should be maximized and invariant ellipsoidal set given in Lemma 2 should be enlarged by adding more penalty on weights of terminal performance index J_1^∞. But this will increase the computational burden of online optimization problem formulated in Theorem 1.

6. A case study

Consider the model of a continuous stirred tank reactor (CSTR), investigated in [45]. Denote the non-zero equilibrium as $\{C_{Aeq}, T_{eq}, T_{ceq}, C_{Afeq}\}$. and define $x = \begin{bmatrix} C_A - C_{Aeq} & T - T_{eq}\end{bmatrix}^T$, $u = T_c - T_{ceq}$, $w = C_{Af} - C_{Afeq}$ and $y = x_2$. Assuming constant fluid volume, the continuous-time linearized dynamics of CSTR for an exothermic, irreversible reaction, $\mathcal{A} \rightarrow \mathcal{B}$, are governed by the state space model in (1) with

$$A = \begin{bmatrix} -\dfrac{q}{V} - k_0 \exp(-\dfrac{E_0}{RT_{eq}}) & -k_0 C_{Aeq}(\dfrac{E_0}{RT_{eq}{}^2})\exp(-\dfrac{E_0}{RT_{eq}}) \\ -\dfrac{(\Delta H)}{d_f C_p} k_0 \exp(-\dfrac{E_0}{RT_{eq}}) & -\dfrac{q}{V} + \left(A\{1,1\}\dfrac{(\Delta H)}{d_f C_p}\right) - \dfrac{UA}{V d_f C_p}\end{bmatrix}$$

Table 1
CSTR parameters.

Parameter	Description	Value
q	Fluid flow rate (l/min)	100
V	Volume of fluid (l)	100
C_{Af}	Final concentration of $\mathcal{A}$ (mol/l)	1
k_0	Arrhenius constant (min^{-1})	7.2×10^{10}
T_f	Final reactor's temperature (K)	350
d_f	Fluid density (g/l)	10^3
C_p	Specific heat of fluid, J/(g K)	0.239
E_0	Activation energy (J/mol)	7.275×10^4
R	Universal gas constant, J/(K mol)	8.3145
ΔH	Specific heat of reaction (J/mol)	-1.2×10^4
UA	Heat transfer constant, J/(min K)	5×10^4
T_c^l	Lower coolant's temperature (K)	328
T_c^u	Upper coolant's temperature (K)	348
T^l	Lower reactor's temperature (K)	340
T^u	Upper reactor's temperature (K)	360
C_A^{eq}	Equilibrium concentration (mol/l)	0.5
T^{eq}	Equilibrium reactor's temperature (K)	350
T_c^{eq}	Equilibrium coolant's temperature (K)	338

$$B=\begin{bmatrix}0\\ \dfrac{UA}{Vd_fC_p}\end{bmatrix},\quad D=\begin{bmatrix}0\\ \dfrac{q}{V}\end{bmatrix},\quad C=\begin{bmatrix}0 & 1\end{bmatrix},\quad E=\begin{bmatrix}0\end{bmatrix}$$

The symbol C_A refers to the concentration of $\mathcal{A}$ in the reactor, T refers to the reactor's temperature, and T_c refers to the temperature of the coolant stream. Table 1 provides the description of system parameters that are used in computing the dynamical model of the above continuous-time process.

The objective is to regulate T by manipulating T_c, satisfying the constraints $T_c^l \le T_c \le T_c^u$. By discretizing the above continuous-time model with sampling period ρ_k, we get the following matrices corresponding to process in (3).

$$A_d(\rho_k)=e^{-1.65\rho_k}\begin{bmatrix}A_d(1,1) & -0.03\sin(1.3\rho_k)\\ 38.62\sin(1.3\rho_k) & A_d(2,2)\end{bmatrix}$$

$$B_d(\rho_k)=\begin{bmatrix}h_1(\rho_k)-0.02\\ h_2(\rho_k)+0.95\end{bmatrix}, D_d(\rho_k)=\begin{bmatrix}h_3(\rho_k)-0.0082\\ h_4(\rho_k)+0.45\end{bmatrix}$$

$$A_d(1,1)=(\cos(1.3\rho_k)-0.27\sin(1.3\rho_k)),$$
$$A_d(2,2)=(\cos(1.3\rho_k)+0.27\sin(1.3\rho_k))$$

$$h_1(\rho_k)=e^{-1.65\rho_k}(0.02\cos(1.3\rho_k)+0.024\sin(1.3\rho_k))$$
$$h_2(\rho_k)=e^{-1.65\rho_k}(-0.95\cos(1.3\rho_k)+0.41\sin(1.3\rho_k))$$
$$h_3(\rho_k)=e^{-1.65\rho_k}(0.0082\cos(1.3\rho_k)+0.0113\sin(1.3\rho_k))$$
$$h_4(\rho_k)=e^{-1.65\rho_k}(-0.45\cos(1.3\rho_k)+0.1985\sin(1.3\rho_k))$$

In this paper, it is assumed that the sampling period ρ_k is varying non-uniformly in the interval

$$0.1\le\rho_k\le0.6$$

We used Theorem 1 to design an observer-based controller and for that we applied the proposed methodology of polytopic approximation to the above time-varying matrices. We get a linear polytopic uncertain system with the following polytopic vertices.

$$A_d^1=\begin{bmatrix}0.8112 & -0.0030\\ 4.2450 & 0.8705\end{bmatrix},\ A_d^2=\begin{bmatrix}0.5777 & -0.0072\\ 10.1081 & 0.7188\end{bmatrix},$$

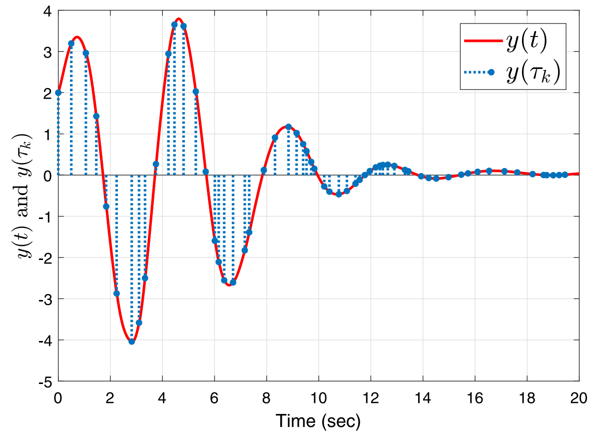

Fig. 4. Output and its non-uniform sampling.

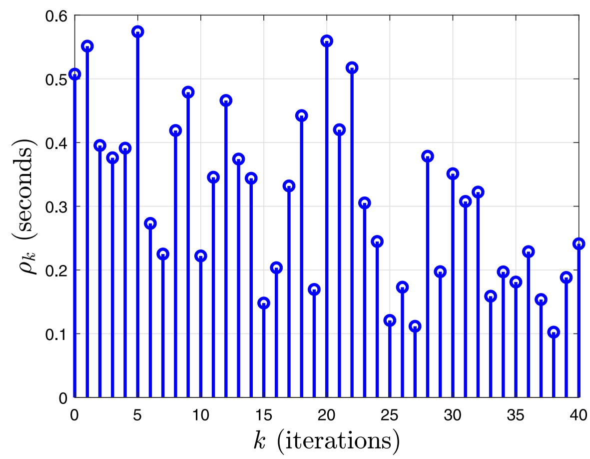

Fig. 5. Sampling period variations.

$$A_d^3=\begin{bmatrix}0.4284 & -0.0030\\ 4.2453 & 0.4877\end{bmatrix},\ A_d^4=\begin{bmatrix}0.1949 & -0.0072\\ 10.1081 & 0.3360\end{bmatrix},$$

$$B_d^1=\begin{bmatrix}-0.0003\\ 0.1956\end{bmatrix},\ B_d^2=\begin{bmatrix}-0.0068\\ -0.0413\end{bmatrix},\ B_d^3=\begin{bmatrix}-0.0003\\ 1.0416\end{bmatrix},$$

$$B_d^4=\begin{bmatrix}-0.0068\\ 0.8047\end{bmatrix},\ D_d^1=\begin{bmatrix}-0.0002\\ 0.0935\end{bmatrix},\ D_d^2=\begin{bmatrix}-0.0033\\ -0.0197\end{bmatrix},$$

$$D_d^3=\begin{bmatrix}-0.0002\\ 0.4979\end{bmatrix},\ D_d^4=\begin{bmatrix}-0.0033\\ 0.3847\end{bmatrix}$$

For the studied system, $\bar{u}=10$ and assume that $e_0\in\mathcal{E}_{S_e}$, where $S_e=\mathrm{diag}\begin{bmatrix}3 & 0.15\end{bmatrix}$, and $x_0\in\mathscr{X}_0=\{[0.5,\ 2]^T+e_0|e_0\in\mathcal{E}_{S_e}\}$. In the simulation, we take $\hat{x}_0=[0.3,\ 4]^T$, $c=0.15$, and $\sigma_k^{i,j}=0.02$ for all $k\ge0$. Choose $\mathcal{W}_x=20I$, $\mathcal{W}_u=1$. Moreover, w_k is randomly generated in the interval $\begin{bmatrix}-1 & 1\end{bmatrix}$ with $w_0=0$. Solving Lemma 1 with fixing the decay rate $s_{\rho_k}^{i,j}=0.02$ at the start and then maximizing as illustrated in Procedure 1, we obtain the state observer gain $L_o=\begin{bmatrix}0.0084 & 0.6183\end{bmatrix}^T$.

Fig. 4 shows the output and its non-uniform sampling. As evident from the figure, the output converges to zero, this means the controller successfully regulates the process temperature around its equilibrium point. Fig. 5 shows the plot of variations in the sampling period. We can see that there is a significant variation in sampling period at different sampling instants. Fig. 6 shows the input $u_k=u_{0|k}$ implemented to the plant. It is evident from the simulation result that the control action u_k satisfies the input constraint (9) and is updated non-uniformly.

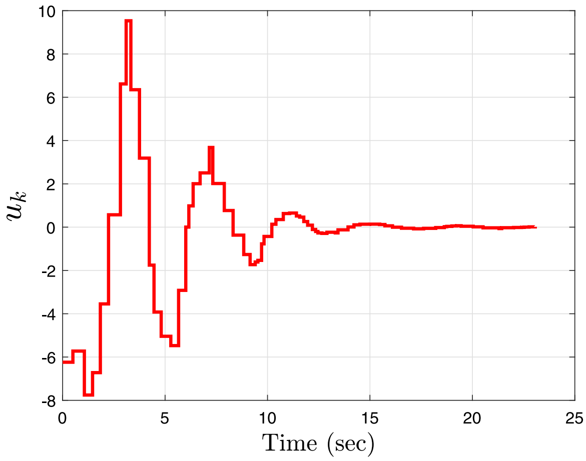

Fig. 6. Control input.

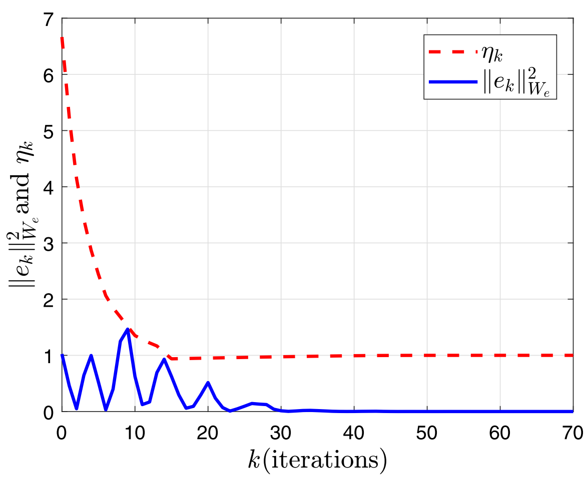

Fig. 7. Estimation error bound and its norm.

Fig. 7 shows the plot of estimation error bound η_k and weighted norm of error e_k. As evident from the figure, the weighted norm of error is less than EEB, which clearly satisfies Assumption 1. The simulation results clearly demonstrate that the proposed design procedure is effective for controlling constrained non-uniformly sampled systems in the presence of bounded disturbances.

7. Conclusion

In this paper, we have proposed an observer-based quasi min–max robust model predictive control for constrained non-uniformly sampled systems in the presence of bounded unknown disturbances. NUSS was formulated into a linear polytopic uncertain system based on variations in the sampling period. An offline state observer was formulated followed by a state feedback controller, to estimate the states and then control the behavior of the system under consideration. The proposed controller guarantees recursive feasibility of the optimization problem and ensures stability of the closed-loop system for all variations of the sampling period. The overall performance was demonstrated through a benchmark problem,which is found to be quite satisfactory. It may be noted that the proposed method relies on polytopic modeling of the closed-loop sampled-data system and the number of vertices of the polytope grow with the size of the system; therefore, the computational time required to solve the online optimization problem will increase. This may limit the applicability of the procedure to very-large-scale system systems. One of our future research focus is to develop a computationally simple model of the non-uniformly sampled system.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Proof of Lemma 1

Since $\|w_k\|^2 \leq 1$, $\Phi_{e,k} \geq 1$ is equivalent to

$\Phi_{e,k} \geq \|w_k\|^2$

Therefore, (13) is equivalent to

$$\Phi_{e,k} \geq \|w_k\|^2 \implies \Phi_{e,k} - \Phi_{e,k+1} \geq 0 \tag{A.1}$$

(A.1) is satisfied iff there exists a real scalar $s(\rho_k) > 0$, i.e a sampling period dependent decay rate for which we have

$$s(\rho_k) = \sum_{i=1}^{n_v}\sum_{j=1}^{n_v} \delta^i(\rho_k)\omega^j(\rho_k)s^{i,j}_{\rho_k}, \quad \sum_{i=1}^{n_v}\sum_{j=1}^{n_v} \delta^i(\rho_k)\omega^j(\rho_k) = 1$$

Now by multiplying $s(\rho_k)$ to both sides of (A.1) and rearranging the terms, we get

$$-s(\rho_k)\Phi_{e,k} + s(\rho_k)w_k^T w_k \leq 0 \tag{A.2}$$

Applying S-procedure on (A.1) and (A.2), we get

$$\Phi_{e,k} - \Phi_{e,k+1} - s(\rho_k)\Phi_{e,k} + s(\rho_k){w_k}^T w_k \geq 0$$

Now by substituting $\Phi_{e,k} = e_k^T W_e e_k$ and applying Schur complement, we obtain

$$\begin{bmatrix} (1-s(\rho_k))W_e & \star & \star \\ 0 & s(\rho_k)I & \star \\ W_eA_d(\rho_k) - W_eL_oC & W_eD_d(\rho_k) - W_eL_oE & W_e \end{bmatrix} \geq 0$$

By change of variables, i.e

Let $Z_e = W_eL_o$, $\implies Z_e^T = L_o^T W_e$, we obtain

$$\begin{bmatrix} (1-s(\rho_k))W_e & \star & \star \\ 0 & s(\rho_k)I & \star \\ W_eA_d(\rho_k) - Z_eC & W_eD_d(\rho_k) - Z_eE & W_e \end{bmatrix} \geq 0 \tag{A.3}$$

The inequality (A.3) then becomes affine with respect to $s(\rho_k)$, $A_d(\rho_k)$ and $D_d(\rho_k)$. It is sufficient to test it at vertices of the polytope, leading to (14). This concludes the proof. □

Appendix B. Proof of Lemma 3

1. By solving Theorem 1 at sampling instant k, it needs to prove that the feasible solution can be formulated by solving the optimization problem at sampling instant $k+1$. Take

$$u_{0|k+1} = u_{1|k} = F_k\hat{x}_{1|k} \tag{B.1a}$$
$$\gamma_{k+1} = \gamma_k \tag{B.1b}$$
$$Q_{k+1} = \vartheta_{k+1}Q_k \tag{B.1c}$$
$$Z_{k+1} = \vartheta_{k+1}Z_k \tag{B.1d}$$

$$X_{k+1} = X_k \tag{B.1e}$$

$$\sigma^{i,j}_{k+1} = \sigma^{i,j}_k \tag{B.1f}$$

$$\beta_{k+1} = \vartheta^{-1}_{k+1}\frac{\eta_{k+2}}{\eta_{k+1}}\beta_k \quad (i.e.,\ \alpha_{k+1} = \alpha_k) \tag{B.1g}$$

where $\vartheta_{k+1} := \frac{\gamma_{k+1}}{\gamma_k}$. Let us check {(9), (16), (17), (19)} at sampling instant $k+1$.

Since $u_{1|k}$ satisfies the input constraint (19), so replacing k by $k+1$ and applying (B.1a), it can be ensured that the input constraint expressed in (9) is feasible. Now by applying congruence transformation on (16), via $\vartheta^{\frac{1}{2}}_{k+1}\mathrm{diag}\{I, \vartheta^{-1}_{k+1}I, \vartheta^{-1}_{k+1}I, I, I, I\}$, and substituting (B.1b)–(B.1d), (B.1f)–(B.1g), it is shown that (16) is feasible for all $k > 0$, $i \in \{1, \ldots, n_v\}$, $j \in \{1, \ldots, n_v\}$.

According to (18), $\eta_{k+2} \le \frac{\tilde{\eta}_{k+2}}{\alpha_k}$, which means

$$\eta_{k+2} \le \frac{\eta_{k+1}}{\beta_k\gamma_k}\gamma_k - (1-\sigma_{1|k})(\|\hat{x}_k\|^2_{\mathcal{W}_x} + \|u_{0|k}\|^2_{\mathcal{W}_u}) - \|\hat{x}_{k+1}\|^2_{\mathcal{W}_x + F_k^T\mathcal{W}_u F_k} - \|\hat{x}_{1|k+1}\|^2_{P_k} \tag{B.2}$$

The above item $-(1-\sigma_{1|k})(\|\hat{x}_k\|^2_{\mathcal{W}_x} + \|u_{0|k}\|^2_{\mathcal{W}_u})$ will be discarded. At sampling instant $k+1$, applying (B.1a)–(B.1c) (leading to $P_{k+1} = P_k$ and $\tilde{x}_{1|k+1} = \hat{x}_{k+2}$) and (B.1g), it is shown that (B.2) satisfied.

$$\|\hat{x}_{k+1}\|^2_{\mathcal{W}_x} + \|u_{0|k+1}\|^2_{\mathcal{W}_u} + \|\hat{x}_{k+2}\|^2_{P_{k+1}} \le (1-\beta_{k+1})\gamma_{k+1} \tag{B.3}$$

Now by changing the subscript $k+1$ in (B.3) by k, one can obtain (27). It should be noted that (17) is guaranteed by (27). Hence, it is proved that (17) is also feasible for sampling instant $k+1$ if and only if it is feasible for sampling instant k.

Now for the input constraint (19) to be feasible for sampling instant $k+1$, replace k by $k+1$ and apply (B.1b)–(B.1e). Hence, (19) is satisfied for $k+1$.

2. (B.1b) is a feasible solution for γ at the respective sampling instant $k+1$ but it satisfies $\gamma_{k+1} \le \gamma_k$, which clearly describes that γ is monotonically decreasing function and thus ensures the robust stability of the closed-loop system.

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